(I)

The Method of Characteristic 2nd order. Wave Equation

IVP. $\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$

Infinite Line

$-\infty < x < \infty,\ t > 0.$

General Soln: $u(x,t) = F(x-ct) + G(x+ct)$

Now, $f(x) = u(x,0) = F(x) + G(x)$ (1)

$g(x) = u_t(x,0) = -cF'(x) + cG'(x).$ (2)

$\frac{d}{dx}$ (1) $\quad [f'(x) = F'(x) + \cancel{G'(x)}]\ (-c)$

$$g(x) = -cF'(x) + c\cancel{G'(x)}$$

$$-cf'(x) + g(x) = -2cF'(x)$$

$$\boxed{F'(x) = \frac{1}{2} f'(x) - \frac{1}{2c} g(x)}$$

$\int_a^x dx$

$$\Rightarrow \boxed{F(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_a^x g(\bar{x})\,d\bar{x} + \underbrace{F(a) - \frac{1}{2} f(a)}_{k}}$$

From (1)

$$\boxed{\begin{aligned} G(x) &= f(x) - F(x) = \\ &= \frac{1}{2} f(x) + \frac{1}{2c} \int_a^x g(\bar{x})\,d\bar{x} - k \end{aligned}}$$

Therefore, $u(x,t) = F(x-ct) + G(x+ct) =$

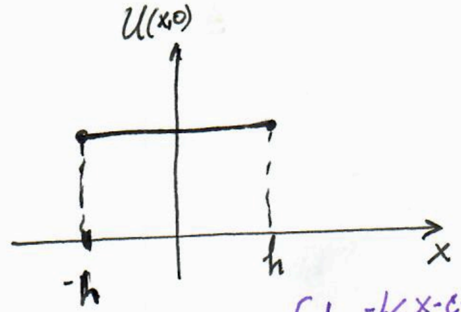
$$= \frac{1}{2} f(x-ct) - \frac{1}{2c} \int_a^{x-ct} g(\bar{x})\,d\bar{x} + \frac{1}{2} f(x+ct) + \frac{1}{2c} \int_a^{x+ct} g(\bar{x})\,d\bar{x}$$

$$\therefore \quad \boxed{u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct) \right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\, d\bar{x}}$$

D'Alembert formula.

Example:- Solve the IVP.

$$\begin{cases} u_{tt} = c^2 u_{xx}, & -\infty < x < \infty, \quad t > 0. \\ u(x,0) = f(x) = \begin{cases} 1, & |x| < h \\ 0, & |x| > h \end{cases} \\ u_t(x,0) = g(x) \equiv 0. \end{cases}$$



D'Alambert's Soln.

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right]}$$

Notice that $f(x-ct) = \begin{cases} 1, & -h < x-ct < h \\ 0, & \text{otherwise} \end{cases}$

$f(x+ct) = \begin{cases} 1, & -h < x+ct < h \\ 0, & \text{otherwise} \end{cases}$

In Region I: $-h < x-ct < h$ and $-h < x+ct < h$

$$u(x,t) = \frac{1}{2}\left[1+1\right] = 1.$$

In Region II: $-h < x-ct < h$ and $x+ct > h$.

$$u(x,t) = \frac{1}{2}\left[\overset{=1}{f(x-ct)} + \overset{=0}{f(x+ct)}\right] = \frac{1}{2}.$$

In Region III: $x-ct < -h$ and $|x+ct| < h$.

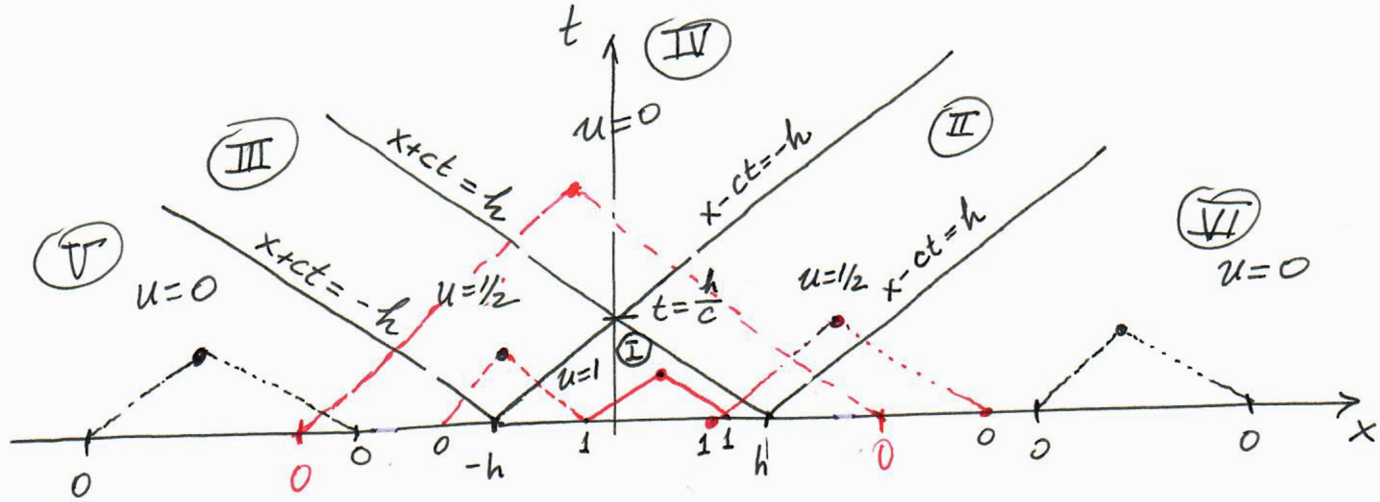
$$u(x,t) = \frac{1}{2}\left[\overset{=0}{f(x-ct)} + \overset{=1}{f(x+ct)}\right] = \frac{1}{2}.$$

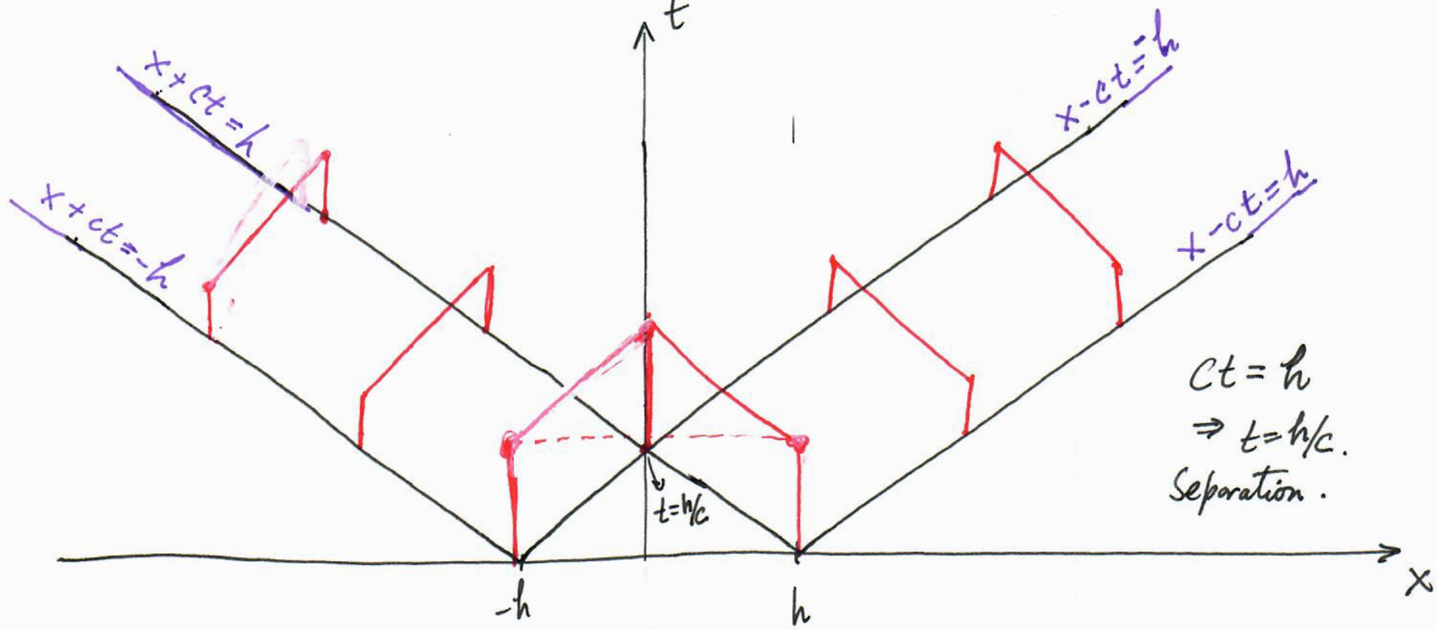
In Region IV $x-ct < -h$ and $x+ct > h$.

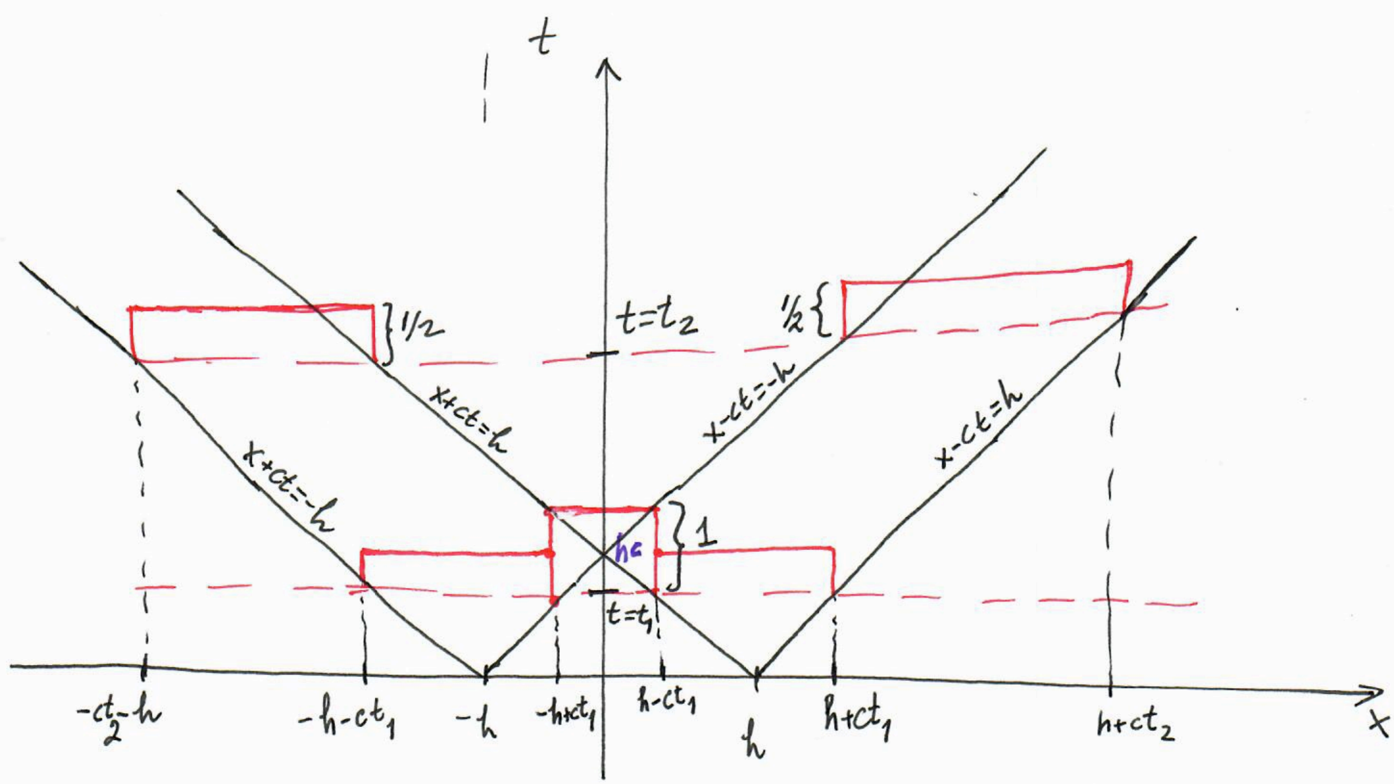
$$u(x,t) = \frac{1}{2}\left[0+0\right] = 0.$$

In Region V: $x-ct < -h$ and $x+ct < -h$. $\Rightarrow u(x,t) = \frac{1}{2}[0+0] = 0$

In Region VI: $x-ct > h$ and $x+ct > h \Rightarrow u(x,t) = \frac{1}{2}[0+0] = 0.$


t
IV
u=0
III
II
x+ct=h
x-ct=-h
V
VI
x+ct=-h
u=1/2
x-ct=h
u=0
u=0
t=h/c
I
u=1
u=1/2
x
0
0
0
0
-h
1
1 1
h
0
0
0
0


t
x+ct=h
x+ct=-h
x-ct=h
x-ct=h
ct=h
⇒ t=h/c.
Separation.
t=h/c
x
-h
h


t
t=t_2
1/2
1/2
x+ct=h
x-ct=-h
x-ct=h
x+ct=-h
h/c
1
t=t_1
x
-ct_2-h
-h-ct_1
-h
-h+ct_1
h-ct_1
h
h+ct_1
h+ct_2